

# Cohomology of rigid curve with semi-stable covering

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## Abstract

We construct a semi-stable formal model of a wide open rigid curve with a semi-stable covering, and study the  $\ell$ -adic cohomology of the rigid curve. We describe the  $\ell$ -adic cohomology of the rigid curve using the  $\ell$ -adic cohomology of the irreducible components of a semi-stable reduction and homology and cohomology of some graphs. We also prove the functoriality of the description for a finite flat morphism that is compatible with semi-stable coverings of wide open rigid curves.

## Introduction

Let  $K$  be a complete discrete valued field with a non-trivial valuation. We assume that the residue field  $k$  of  $K$  is an algebraic extension of a finite field. We consider a wide open rigid curve  $W$  over  $K$  with a semi-stable covering. The notion of a semi-stable covering of a wide open rigid curve is due to Coleman-McMurdy (cf. [Co], [CM]). A semi-stable covering of a wide open rigid curve is an analogy of a semi-stable model of an algebraic curve. In this paper, we construct a semi-stable formal model of  $W$  from the semi-stable covering of  $W$ .

Let  $\ell$  be a prime number different from the characteristic of  $k$ . The purpose of this paper is to study the  $\ell$ -adic cohomology of  $W$ . Let  $\mathscr{W}$  be a semi-stable formal model of  $W$  constructed from the semi-stable covering of  $W$ , and  $\Gamma_{\mathscr{W}}$  be the dual graph of the geometric closed fiber of  $\mathscr{W}$ . In this paper, we describe the  $\ell$ -adic cohomology of  $W$  using the  $\ell$ -adic cohomology of the irreducible components of the geometric closed fiber of  $\mathscr{W}$  and homology and cohomology of some variants of  $\Gamma_{\mathscr{W}}$ .

We also study a relative situation. Let  $W_1, W_2$  be wide open rigid curves with semi-stable coverings. We consider a finite flat morphism  $f: W_1 \rightarrow W_2$  that is compatible with the semi-stable coverings. We show that such a morphism extends to a morphism between their formal semi-stable models. The pushforward and the pullback on the  $\ell$ -adic cohomology by  $f$  induce morphisms on the  $\ell$ -adic cohomology of the irreducible components of the geometric closed fibers and homology and cohomology of the graphs. The induced morphism on the  $\ell$ -adic cohomology of the irreducible components of the geometric closed fibers is a usual one. We will describe the morphisms on homology and cohomology of the graphs using the induced morphisms on graphs. This description essentially gives a functoriality of the weight spectral sequence for  $f$ , which is not necessarily algebraizable.

Lubin-Tate spaces for  $GL_2$  are examples of wide open rigid curves. The intention of our research is in the application to the study of action of Hecke operators on the  $\ell$ -adic cohomology of Lubin-Tate spaces for  $GL_2$ .

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## Notation

Throughout this paper, we use the following notation. For a field  $L$  with a non-trivial non-archimedean valuation, the ring of integers of  $L$  is denoted by  $\mathcal{O}_L$ . For a field  $F$ , the algebraic closure of  $F$  is denoted by  $\overline{F}$ . For a vector space  $V$  over a field  $F$ , the dual vector space of  $V$  over  $F$  is denoted by  $V^*$ . For a commutative ring  $A$ , a commutative  $A$ -algebra  $B$  and a scheme  $X$  over  $A$ , the base change of  $X$  to  $B$  is denoted by  $X_B$ . For an extension  $L_2$  over  $L_1$  of fields with nontrivial non-archimedean complete valuations and a rigid space  $X$  over  $L_1$ , the base change of  $X$  to  $L_2$  is denoted by  $X_{L_2}$ .

## 1 Semi-stable covering

In this section, we recall the notion of a semi-stable covering and some related results from [CM].

Let  $K$  be a complete discrete valued field with a non-trivial valuation  $v$ . We assume that the residue field  $k$  of  $K$  is an algebraic extension of a finite field. We normalize the valuation so that  $v(K^\times) = \mathbb{Z}$ . We put  $v(0) = +\infty$  and  $|x| = p^{-v(x)}$  for  $x \in K$ . The ring of integers of  $K$ , the maximal ideal of  $\mathcal{O}_K$  and the residue field of  $K$  are denoted by  $\mathfrak{m}_K$  and  $k$  respectively. Let  $\mathbf{C}$  be the completion of an algebraic closure of  $K$ . The absolute value  $|\cdot|$  on  $K$  naturally extends to  $|\cdot|$  on  $\mathbf{C}$ .

For  $r \in |\mathbf{C}^\times|$ , let  $B_K[r]$  and  $B_K(r)$  be the rigid spaces over  $K$  whose  $\mathbf{C}$ -valued points are  $\{x \in \mathbf{C} \mid |x| \leq r\}$  and  $\{x \in \mathbf{C} \mid |x| < r\}$ , which we call a closed disk and an open disk respectively. For  $r, s \in |\mathbf{C}^\times|$  satisfying  $r \leq s$ , let  $A_K[r, s]$  and  $A_K(r, s)$  be the rigid spaces over  $K$  whose  $\mathbf{C}$ -valued points are  $\{x \in \mathbf{C} \mid r \leq |x| \leq s\}$  and  $\{x \in \mathbf{C} \mid r < |x| < s\}$ , which we call a closed annulus and an open annulus respectively. The widths of  $A_K[r, s]$  and  $A_K(r, s)$  are defined by  $\log_p(s/r)$ . A closed annulus of the form  $A_K[r, r]$  for  $r \in |\mathbf{C}^\times|$  is called a circle. For such  $r, s$ ,  $A_K[r, s]$  and  $A_K(r, s)$  are defined similarly, which we call semi-open annuli.

**Definition 1.1.** *A wide open rigid curve over  $K$  is a one-dimensional smooth rigid space  $W$  over  $K$  which contains affinoid subdomains  $X$  and  $Y$  such that*

1.  $W \setminus X$  is a disjoint union of finitely many open annuli,
2.  $X$  is relatively compact in  $Y$ ,
3.  $Y \cap V$  is a semi-open annulus for all connected components  $V$  of  $W \setminus X$ .

We call  $X$  an underlying affinoid of  $W$ .

**Theorem 1.2.** [CM, Theorem 2.18] *Let  $W$  be a wide open rigid curve over  $K$  with an underlying affinoid  $X$ . Then  $W$  may be completed to a proper algebraic curve  $C$  over  $K$  by gluing closed disks to the connected components of  $W \setminus X$ .*

For a rigid space  $X$  over  $K$ , we put

$$|f|_{\sup} = \sup_{x \in X(\mathbf{C})} |f(x)|$$

for  $f \in \mathcal{O}_X(X)$ . For an affinoid rigid space  $X$ ,

$$\begin{aligned} A^\circ(X) &= \{f \in \mathcal{O}_X(X) \mid |f|_{\sup} \leq 1\} \\ A^{\circ\circ}(X) &= \{f \in \mathcal{O}_X(X) \mid |f(x)| < 1 \text{ for all } x \in X(\mathbf{C})\}, \end{aligned}$$

and we consider  $A^\circ(X)$  as a linearly topologized ring with the ideal of definition  $A^{\circ\circ}(X)$ .

For an affinoid rigid space  $X$ , we put

$$\overline{X} = \text{Spec}(A^\circ(X)/A^{\circ\circ}(X)),$$

which we call the reduction of  $X$ . The canonical reduction map of an affinoid rigid space  $X$  is denoted by  $\text{Red}_X: X \rightarrow \overline{X}$ .

**Definition 1.3.** *A basic wide open pair of rigid curves is a pair  $(W, X)$  where  $W$  is a connected wide open rigid curve over  $K$  and  $X$  is an underlying affinoid  $X$  such that*

1.  $\overline{X}$  is an irreducible curve with at worst ordinary double points as singularities,
2.  $\{|f|_{\sup} \mid f \in \mathcal{O}_X(X)\} = |K|$ ,
3. the components of  $W \setminus X$  are isomorphic to annuli of the form  $A_K(r, 1)$  for  $r \in |K|^\times$  satisfying  $r < 1$ .

We say that  $W$  is a basic wide open rigid curve, if  $(W, X)$  is a basic wide open pair for some  $X$ .

If  $(W, X)$  is a basic wide open pair of rigid curves, then  $\overline{X}^\circ$  denotes the compactification of  $\overline{X}$  that is smooth at cusps.

**Definition 1.4.** *Let  $C$  be a wide open rigid curve or a proper smooth curve over  $K$ . A semi-stable covering of  $C$  over  $K$  is a finite set  $\mathcal{S}$  of basic wide open pairs  $(U, U^u)$  such that*

1.  $\mathcal{S}^w = \{U \mid (U, U^u) \in \mathcal{S}\}$  is an admissible covering of  $C$ ,
2. if  $U_1, U_2 \in \mathcal{S}^w$  and  $U_1 \neq U_2$ , then  $U_1 \cap U_2$  is a disjoint union of connected components of  $U_1 \setminus U_1^u$ ,
3. if  $U_1, U_2$  and  $U_3$  are three distinct elements of  $\mathcal{S}^w$ , then  $U_1 \cap U_2 \cap U_3 = \emptyset$ .

**Theorem 1.5.** *[CM, Theorem 2.36] Let  $C$  be a proper smooth curve over  $K$ . If  $C$  has a semi-stable covering over  $K$ , then  $C$  has an associated semi-stable model over  $\mathcal{O}_K$  whose reductions have at least two components.*

## 2 Morphism between graphs

In this paper, a graph means a finite ordered graph such that each edge has two order.

Let  $\Gamma$  be a graph. The set of the vertices of  $\Gamma$  is denoted by  $\mathcal{V}(\Gamma)$ , and the set of the ordered edges of  $\Gamma$  is denoted by  $\mathcal{E}(\Gamma)$ . For  $e \in \mathcal{E}(\Gamma)$ , the source of  $e$  and the target of  $e$  are denoted by  $s(e)$  and  $t(e)$  respectively. For  $e \in \mathcal{E}(\Gamma)$ , let  $\bar{e}$  be the ordered edge obtained by reversing the order of  $e$ .

We take a prime number  $\ell$  that is different from the characteristic of  $k$ . Let  $V(\Gamma, \mathbb{Q}_\ell)$  be the  $\mathbb{Q}_\ell$ -vector space generated by  $\mathcal{V}(\Gamma)$ , and  $E(\Gamma, \mathbb{Q}_\ell)$  be the  $\mathbb{Q}_\ell$ -vector space generated by  $\mathcal{E}(\Gamma)$  with relation  $e = -\bar{e}$  for all  $e \in \mathcal{E}(\Gamma)$ . We consider two  $\mathbb{Q}_\ell$ -linear maps

$$\begin{aligned} d: E(\Gamma, \mathbb{Q}_\ell) &\longrightarrow V(\Gamma, \mathbb{Q}_\ell); \quad e \mapsto t(e) - s(e), \\ \delta: V(\Gamma, \mathbb{Q}_\ell) &\longrightarrow E(\Gamma, \mathbb{Q}_\ell); \quad v \mapsto \sum_{e \in \mathcal{E}(\Gamma), t(e)=v} e. \end{aligned}$$

We put  $H_1(\Gamma, \mathbb{Q}_\ell) = \text{Ker}(d)$  and  $H^1(\Gamma, \mathbb{Q}_\ell) = \text{Coker}(\delta)$ . A cycle  $R$  in  $\Gamma$  can be considered as an element of  $H_1(\Gamma, \mathbb{Q}_\ell)$ . We consider a natural bilinear pairing

$$\langle \cdot, \cdot \rangle: E(\Gamma, \mathbb{Q}_\ell) \times E(\Gamma, \mathbb{Q}_\ell) \longrightarrow \mathbb{Q}_\ell$$

determined by

$$\langle e_1, e_2 \rangle = \begin{cases} 1 & \text{if } e_1 = e_2 \\ 0 & \text{if } e_1 \neq e_2, \bar{e}_2 \end{cases}$$

for  $e_1, e_2 \in \mathcal{E}(\Gamma)$ . Then this pairing induces a bilinear perfect pairing

$$H_1(\Gamma, \mathbb{Q}_\ell) \times H^1(\Gamma, \mathbb{Q}_\ell) \longrightarrow \mathbb{Q}_\ell.$$

Therefore we have a canonical isomorphism  $H^1(\Gamma, \mathbb{Q}_\ell) \cong (H_1(\Gamma, \mathbb{Q}_\ell))^*$ .

Let  $\Gamma_1, \Gamma_2$  be graphs.

**Definition 2.1.** A finite flat morphism  $\phi: \Gamma_1 \rightarrow \Gamma_2$  of degree  $n$  consists of the following data:

- Surjective maps  $\phi_V: \mathcal{V}(\Gamma_1) \rightarrow \mathcal{V}(\Gamma_2)$  and  $\phi_E: \mathcal{E}(\Gamma_1) \rightarrow \mathcal{E}(\Gamma_2)$  satisfying  $\phi_V \circ s = s \circ \phi_E$  and  $\phi_V \circ t = t \circ \phi_E$ .
- Positive integers  $n_v$  and  $n_e$  for all  $v \in \mathcal{V}(\Gamma_1)$  and  $e \in \mathcal{E}(\Gamma_1)$  such that  $n_e = n_{\bar{e}}$  for  $e \in \mathcal{E}(\Gamma_1)$ , and the map  $s: \phi_E^{-1}(e') \rightarrow \phi_V^{-1}(s(e'))$  is surjective,

$$\sum_{e \in \phi_E^{-1}(e')} n_e = n \quad \text{and} \quad \sum_{e \in \phi_E^{-1}(e'), s(e)=v} n_e = n_v$$

for  $e' \in \mathcal{E}(\Gamma_2)$  and  $v \in \phi_V^{-1}(s(e'))$ .

Let  $\phi: \Gamma_1 \rightarrow \Gamma_2$  be a finite flat morphism of degree  $n$ . For a cycle  $R = e_1 \cdots e_m$  in  $\Gamma_1$ , the cycle  $\phi_E(e_1) \cdots \phi_E(e_m)$  in  $\Gamma_2$  is denoted by  $\phi(R)$ .

**Proposition 2.2.** Let  $R'$  be a cycle in  $\Gamma_2$ . Then there are cycles  $R_1, \dots, R_m$  in  $\Gamma_1$  such that  $\phi(R_i) = R'$  for  $1 \leq i \leq m$  and

$$|\{1 \leq i \leq m \mid e \in \mathcal{E}(R_i)\}| = n_e$$

for all  $e \in \mathcal{E}(\Gamma_1)$  satisfying  $\phi_E(e) \in \mathcal{E}(R')$ . Furthermore,  $\sum_{i=1}^m R_i \in H_1(\Gamma_1, \mathbb{Q}_\ell)$  does not depend on a choice of  $R_1, \dots, R_m$ .

*Proof.* By replacing  $\Gamma_1, \Gamma_2$  by their subgraphs, we may assume  $\Gamma_2 = R'$ .

We prove the first claim by induction on  $n$ . If  $n = 1$ , the claim is trivial. We assume  $n \geq 2$ . By surjectivity of  $s: \phi_E^{-1}(e') \rightarrow \phi_V^{-1}(s(e'))$  for  $e' \in \mathcal{E}(\Gamma_2)$ , we can easily find a cycle  $R$  in  $\Gamma_1$  such that  $\phi(R) = R'$ . Then we put

$$n'_v = \begin{cases} n_v - 1 & \text{if } v \in \mathcal{V}(R) \\ n_v & \text{if } v \notin \mathcal{V}(R) \end{cases} \text{ and } n'_e = \begin{cases} n_e - 1 & \text{if } e \in \mathcal{E}(R) \\ n_e & \text{if } e \notin \mathcal{E}(R) \end{cases}$$

for  $v \in \mathcal{V}(\Gamma_1)$  and  $e \in \mathcal{E}(\Gamma_1)$ . We consider a subgraph  $\Gamma'_1$  of  $\Gamma_1$  obtained from  $\Gamma_1$  by removing  $v \in \mathcal{V}(\Gamma_1)$  and  $e \in \mathcal{E}(\Gamma_1)$  such that  $n'_v = 0$  and  $n'_e = 0$ . We define a positive integer  $\deg(R/R')$  by

$$\phi(R) = \deg(R/R')R' \in H_1(\Gamma_2, \mathbb{Q}_\ell).$$

Then the restriction of  $\phi$  gives a finite flat morphism  $\phi': \Gamma'_1 \rightarrow \Gamma_2$  of degree  $n - \deg(R/R')$  and it suffice to show the claim for  $\phi'$ . This follows from the induction hypothesis.

The last claim follows from the condition  $|\{1 \leq i \leq m \mid e \in \mathcal{E}(R_i)\}| = n_e$  for all  $e \in \mathcal{E}(\Gamma_1)$ .  $\square$

We define  $\mathbb{Q}_\ell$ -linear maps  $\phi_*: H_1(\Gamma_1, \mathbb{Q}_\ell) \rightarrow H_1(\Gamma_2, \mathbb{Q}_\ell)$  and  $\phi^*: H_1(\Gamma_2, \mathbb{Q}_\ell) \rightarrow H_1(\Gamma_1, \mathbb{Q}_\ell)$  by  $\phi_*(R) = \phi(R)$  for a cycle  $R$  in  $\Gamma_1$  and  $\phi^*(R') = \sum_{i=1}^m R_i$  for a cycle  $R'$  in  $\Gamma_2$ , where  $R_1, \dots, R_m$  are as in Proposition 2.2.

We define  $\phi_*: H^1(\Gamma_1, \mathbb{Q}_\ell) \rightarrow H^1(\Gamma_2, \mathbb{Q}_\ell)$  and  $\phi^*: H^1(\Gamma_2, \mathbb{Q}_\ell) \rightarrow H^1(\Gamma_1, \mathbb{Q}_\ell)$  as the dual  $\mathbb{Q}_\ell$ -linear maps of  $\phi^*: H_1(\Gamma_2, \mathbb{Q}_\ell) \rightarrow H_1(\Gamma_1, \mathbb{Q}_\ell)$  and  $\phi_*: H_1(\Gamma_1, \mathbb{Q}_\ell) \rightarrow H_1(\Gamma_2, \mathbb{Q}_\ell)$  respectively.

### 3 Formal model of rigid curve

Let  $W$  be a connected wide open rigid curve with a semi-stable covering  $\mathcal{S} = \{(U_i, U_i^\natural) \mid i \in I\}$  over  $K$ . In this section, we construct a formal model of  $W$  from the semi-stable covering  $\mathcal{S}$ . First, we recall some facts from [CM].

**Proposition 3.1.** [CM, Proposition 2.21] *Let  $C$  be a proper smooth curve over  $K$ . Let  $L$  be a finite Galois extension of  $K$ , and  $T$  be a finite nonempty Galois stable subset of  $C(L)$ . Suppose  $\{D_t \mid t \in T\}$  is a Galois stable collection of disjoint open disks over  $L$  in  $C$  such that  $D_t \cap T = \{t\}$  for all  $t \in T$ . Then  $C \setminus (\bigcup_{t \in T} D_t)$  is a one-dimensional affinoid over  $K$ .*

**Lemma 3.2.** [CM, Lemma 2.24] *Let  $f: X \rightarrow Y$  be a morphism between smooth one-dimensional affinoid rigid spaces over  $K$ . We assume that  $\overline{X}$  is irreducible.*

- (i) *If  $\overline{f}: \overline{X} \rightarrow \overline{Y}$  is a surjection, then  $f$  is a surjection.*
- (ii) *If  $\overline{f}(\overline{X}) \subset \overline{Y}$  is an open affine subset and  $f: X(\mathbb{C}) \rightarrow Y(\mathbb{C})$  is an injection, then  $\overline{f}$  is an injection.*

**Lemma 3.3.** [CM, Proposition 2.8] *Let  $X$  be a smooth one-dimensional affinoid rigid curve such that  $||A(X)|| = |K|$ . We assume that  $P \in \overline{X}(\overline{k})$  and  $\text{Red}^{-1}(P)$  is isomorphic to an open annulus over  $K$ . Then  $\overline{X}$  is semi-stable at  $P$ .*

For  $i \in I$ , let  $\{V_j \mid j \in J_i\}$  be the set of connected components of  $U_i \setminus (U_i^\natural \cup \bigcup_{i' \in I, i' \neq i} U_{i'})$ . For different  $i_1, i_2 \in I$ , let  $\{V_j \mid j \in J_{i_1, i_2}\}$  be the set of connected components of  $U_{i_1} \cap U_{i_2}$ . For

$i \in I$  and  $j \in J_i$ , let  $V_j^s$  be a semi-open annulus obtained by adding a circle  $C_j$  to  $V_j$  from the opposite side to  $U_i^u$ . We define a rigid space  $W'$  as  $W \cup \bigcup_{i \in I, j \in J_i} V_j^s$ . We put  $\tilde{I} = I \cup \bigcup_{i \in I} J_i$ , and

$$(U_i, U_i^u) = \begin{cases} (U_i, U_i^u) & \text{if } i \in I \\ (V_{i'}, C_{i'}) & \text{if } i' \in I \text{ and } i \in J_{i'} \end{cases}$$

for  $i \in \tilde{I}$ . We put

$$J_{i_1, i_2} = \begin{cases} i_1 & \text{if } i_2 \in I \text{ and } i_1 \in J_{i_2} \\ i_2 & \text{if } i_1 \in I \text{ and } i_2 \in J_{i_1} \\ \emptyset & \text{otherwise} \end{cases}$$

for different  $i_1, i_2 \in \tilde{I}$ .

**Proposition 3.4.** *For  $i \in \tilde{I}$ , let  $S_i$  be a nonempty  $G_K$ -stable collection of points of the smooth locus of  $\overline{U}_i^u$ . We put  $U(S_i) = U_i^u \setminus (\bigcup_{s \in S_i} \text{Red}_{U_i^u}^{-1}(s))$  for  $i \in \tilde{I}$  and  $X_j(S_{i_1}, S_{i_2}) = U(S_{i_1}) \cup U(S_{i_2}) \cup V_j$  for different  $i_1, i_2 \in \tilde{I}$  and  $j \in J_{i_1, i_2}$ . Then  $U(S_i)$  and  $X_j(S_{i_1}, S_{i_2})$  are affinoid rigid spaces over  $K$ , and  $\text{Red}_{X_j(S_{i_1}, S_{i_2})}$  is compatible with  $\text{Red}_{U(S_{i_1})}$  and  $\text{Red}_{U(S_{i_2})}$  for  $i \in \tilde{I}$  and different  $i_1, i_2 \in \tilde{I}$ .*

*Proof.* We put  $W'(\{S_i\}_{i \in \tilde{I}}) = W' \setminus (\bigcup_{s \in S_i, i \in \tilde{I}} \text{Red}_{U_i^u}^{-1}(s))$ . Let  $C$  be a proper smooth curve over  $K$  obtained from  $W$  as in Theorem 1.2. Then  $W'(\{S_i\}_{i \in \tilde{I}})$  is obtained from  $C$  removing a  $G_K$ -stable union of disjoint open disks. Therefore  $W'(\{S_i\}_{i \in \tilde{I}})$  is an affinoid rigid space over  $K$  by Proposition 3.1.

We take  $i \in \tilde{I}$ . We know that  $U(S_i)$  is affinoid rigid spaces by [FvdP, Lemma 4.8.1]. The natural inclusion map  $j_{U(S_i)}: U(S_i) \rightarrow W'(\{S_i\}_{i \in \tilde{I}})$  induces a map on the reduction  $\bar{j}_{U(S_i)}: \overline{U}(S_i) \rightarrow \overline{W}'(\{S_i\}_{i \in \tilde{I}})$ . Let  $\text{Im}(\bar{j}_{U(S_i)})$  be the image of  $\bar{j}_{U(S_i)}$ . Then  $\text{Im}(\bar{j}_{U(S_i)})$  is a point or an affine open subset of  $\overline{W}'(\{S_i\}_{i \in \tilde{I}})$ .

We assume that  $\text{Im}(\bar{j}_{U(S_i)})$  is a point  $P$ . Then  $\text{Red}_{W'(\{S_i\}_{i \in \tilde{I}})}^{-1}(P)$  is not connected to  $\bigcup_{s \in S_i} \text{Red}_{U_i^u}^{-1}(s)$ . Therefore  $U(S_i) \subset \text{Red}_{W'(\{S_i\}_{i \in \tilde{I}})}^{-1}(P)$  is not connected to  $\bigcup_{s \in S_i} \text{Red}_{U_i^u}^{-1}(s)$ . This is a contradiction. Thus we have proved that  $\text{Im}(\bar{j}_{U(S_i)})$  is an affine open subset of  $\overline{W}'(\{S_i\}_{i \in \tilde{I}})$ . Then the map  $\bar{j}_{U(S_i)}$  is an injection by Lemma 3.2 (ii). Further, we have that  $U(S_i) = \text{Red}_{W'(\{S_i\}_{i \in \tilde{I}})}^{-1}(\text{Im}(\bar{j}_{U(S_i)}))$  by Lemma 3.2 (i). Let  $Y(S_i)$  be the irreducible component of  $\overline{W}'(\{S_i\}_{i \in \tilde{I}})$  that contains  $\text{Im}(\bar{j}_{U(S_i)})$ .

We take  $j \in J_{i_1, i_2}$  for different  $i_1, i_2 \in \tilde{I}$ . Then  $V_j = \text{Red}_{W'(\{S_i\}_{i \in \tilde{I}})}^{-1}(y_j)$  for a point  $y_j$  of  $\overline{W}'(\{S_i\}_{i \in \tilde{I}})$  by the connectedness of  $V_j$ . Further, we have that  $y_j \in Y(S_{i_1}) \cap Y(S_{i_2})$ , because  $V_j$  is connected to  $U(S_{i_1})$  and  $U(S_{i_2})$ .

Therefore we have

$$Y(S_i) = \text{Im}(\bar{j}_{U(S_i)}) \cup \bigcup_{j \in J_{i, i'}, i' \in \tilde{I}} y_j$$

for all  $i \in \tilde{I}$ . Then  $X_j(S_{i_1}, S_{i_2}) = \text{Red}_{W'(\{S_i\}_{i \in \tilde{I}})}^{-1}(\text{Im}(\bar{j}_{U(S_{i_1})}) \cup \text{Im}(\bar{j}_{U(S_{i_2})}) \cup y_j)$  is an affinoid rigid space by [FvdP, Lemma 4.8.1]. The compatibility also follows from [FvdP, Lemma 4.8.1].  $\square$

For a formal scheme  $\mathcal{X}$  over  $\text{Spf } \mathcal{O}_K$ , the closed fiber  $\mathcal{X}_k$  of  $\mathcal{X}$  means the underlying reduced scheme of the ringed space  $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{I})$ , where  $\mathcal{I}$  is an ideal of definition of  $\mathcal{X}$ .

**Theorem 3.5.** *Let  $W$  be a connected wide open rigid curve with a semi-stable covering  $\mathcal{S} = \{(U_i, U_i^u) \mid i \in I\}$  over  $K$ . Then  $W$  has an associated semi-stable formal model over  $\mathcal{O}_K$ .*

*Proof.* We take  $S_i$  for  $i \in \tilde{I}$  as in Proposition 3.4. Then  $\{U_i^u\}_{i \in \tilde{I}} \cup \{X_j(S_{i_1}, S_{i_2})\}_{i_1, i_2 \in \tilde{I}, j \in J_{i_1, i_2}}$  is a pure affinoid covering in the sense of [FvdP, Definition 4.8.3] by Lemma 3.4. This covering gives a formal model  $\mathcal{W}'$  of  $W'$ . The closed fiber  $\mathcal{W}'_k$  is a semi-stable curve by Lemma 3.3.

Let  $Y$  be a reduced closed subscheme of  $\mathcal{W}'_k$  determined by  $\bigcup_{i \in I} \overline{U}_i^{u, c}$ . We define  $\mathcal{W}$  as the formal completion of  $\mathcal{W}'$  along  $Y$ . Then  $\mathcal{W}$  is a formal model of  $W$ . We can check that  $\mathcal{W}$  is independent of a choice of  $S_i$  for  $i \in \tilde{I}$ .  $\square$

We use the notation in the proof of Theorem 3.5. The closed fiber  $\mathcal{W}_k$  of  $\mathcal{W}$  is  $Y$ . The irreducible components of the geometric closed fiber  $\mathcal{W}'_k$  of  $\mathcal{W}'$  consists of proper curves that are also irreducible components of  $Y_k$  and affine schemes that are isomorphic to  $\text{Spec } \overline{k}[X]$ .

**Proposition 3.6.** *For  $i_1 \in I$  and  $i_2 \in J_{i_1}$ , we put  $X'_{i_2}(S_{i_1}) = U(S_{i_1}) \cup V_{i_2}$ . Then  $\mathcal{W}$  is obtained also by patching  $\text{Spf } A^\circ(U_i^u)$  for  $i \in I$ ,  $\text{Spf } A^\circ(X_j(S_{i_1}, S_{i_2}))$  for  $i_1, i_2 \in I$  and  $j \in J_{i_1, i_2}$ , and  $\text{Spf } A^\circ(X'_{i_2}(S_{i_1}))$  for  $i_1 \in I$  and  $i_2 \in J_{i_1}$ .*

*Proof.* This follows from [deJ, Theorem 7.4.1].  $\square$

Let  $C$  be the proper smooth curve  $C$  over  $K$  obtained from  $W$  as in Theorem 1.2. Let  $\{D_{i'}\}_{i' \in I'}$  be the set of the connected components of  $D = C \setminus W$ . For  $i' \in I'$ , there uniquely exists  $i \in I$  and  $j \in J_i$  such that the union of  $D_{i'}$  and  $V_j$  defines an open disk in  $C$ , which is denoted by  $U_{i'}$ . Then  $\mathcal{S}' = \mathcal{S} \cup \{(U_{i'}, D_{i'}) \mid i' \in I'\}$  is a semi-stable covering of  $C$ . Let  $\mathcal{C}$  be a semi-stable model of  $C$  associated to the semi-stable covering  $\mathcal{S}'$  by Theorem 1.5. Then  $Y$  is naturally considered as a closed subscheme of the special fiber  $\mathcal{C}_k$  of  $\mathcal{C}$ .

**Proposition 3.7.** *The formal completion of  $\mathcal{C}$  along  $Y$  is naturally isomorphic to  $\mathcal{W}$ .*

*Proof.* This follows from the construction.  $\square$

**Remark 3.8.** *We have two another construction of  $\mathcal{W}$  as in Proposition 3.6 and Proposition 3.7, and the both construction is important. The construction in Proposition 3.6 implies that a finite flat morphism between wide open rigid curves which is compatible with their semi-stable coverings naturally extends to a morphism between their semi-stable formal model. This fact is very non-trivial from the construction in Proposition 3.7, because such a morphism does not extend a morphism between their compactifications in general. On the other hand, by Proposition 3.7, we see that  $W$  satisfies the condition of [Fa, Proposition 5.9.4].*

## 4 Cohomology of rigid curve

We put  $\mathcal{W}_{\mathcal{O}_K} = \mathcal{W} \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_K$ . The formal nearby cycle functor  $R\Psi_{\mathcal{W}_{\mathcal{O}_K}}$  of  $\mathcal{W}_{\mathcal{O}_K}$  is defined in [Be2, section 2].

Let  $\Gamma_{\mathcal{W}}$  and  $\Gamma_{\mathcal{W}'}$  be dual graphs of  $Y_k$  and  $\mathcal{W}'_k$  respectively (cf. [Li, Definition 10.3.17]). Then  $\Gamma_{\mathcal{W}}$  is a subgraph of  $\Gamma_{\mathcal{W}'}$ . For  $v \in \mathcal{V}(\Gamma_{\mathcal{W}'})$ , let  $Y_v$  be the irreducible component of  $\mathcal{W}'_k$  corresponding to  $v$ ,  $\tilde{Y}_v$  be the normalization of  $Y_v$ , and  $\pi_v: \tilde{Y}_v \rightarrow \mathcal{W}'_k$  be a natural morphism.

**Proposition 4.1.** *Let  $\Lambda$  be a torsion local finite  $\mathbb{Z}_\ell$ -algebra. Then there are canonical isomorphisms*

$$R^0\Psi_{\mathcal{W}_{\circ\mathbf{C}}}\Lambda \cong \Lambda \quad (4.1)$$

$$R^1\Psi_{\mathcal{W}_{\circ\mathbf{C}}}\Lambda \cong \left( \left( \bigoplus_{v \in \mathcal{V}(\Gamma_{\mathcal{W}'})} \pi_{v*}\Lambda \right) / \Lambda \right)^* (-1). \quad (4.2)$$

*Proof.* The isomorphism (4.1) follows from the definition. By [Be2, Theorem 3.1], we have an isomorphism

$$R^1\Psi_{\mathcal{W}_{\circ\mathbf{C}}}\Lambda \cong (R^1\Psi_{\mathcal{C}_{\circ\overline{K}}}\Lambda)|_{Y_{\overline{K}}}.$$

Hence it suffice to prove the isomorphism (4.2) locally at singular points of  $\mathcal{W}'_{\overline{K}}$ .

Let  $x_e$  be a singular points of  $\mathcal{W}'_{\overline{K}}$  corresponding to  $e \in \mathcal{E}(\Gamma_{\mathcal{W}'})$ . Then the formal completion  $\mathcal{W}'_e$  of  $\mathcal{C}_{\circ\overline{K}}$  at  $x_e$  is isomorphic to  $\mathrm{Spf} \mathcal{O}_{\mathbf{C}}[[S, T]]/(ST - c)$  for some  $c \neq 0 \in \mathfrak{m}_K$ . Further,  $\mathrm{Spf} \mathcal{O}_{\mathbf{C}}[[S, T]]/(ST - c)$  is isomorphic to the formal completion of  $\mathcal{X} = \mathrm{Spec} \mathcal{O}_{\overline{K}}[S, T]/(ST - c)$  at a point  $x_0$  of the special fiber defined by  $S = T = 0$ . Then we have isomorphisms

$$(R^1\Psi_{\mathcal{C}_{\circ\overline{K}}}\Lambda)|_{x_e} \cong R^1\Psi_{\mathcal{W}'_e}\Lambda \cong (R^1\Psi_{\mathcal{X}}\Lambda)|_{x_0} \cong H^1(\mathbb{G}_{m, \overline{K}}, \Lambda)$$

by [Be2, Theorem 3.1] and [SGA7II, Exposé XV Proposition 2.2.3]. Let  $i_0$  and  $i_\infty$  be the closed immersion of the zero point and the infinity point into  $\mathbb{P}^1$  respectively. Then  $H^1(\mathbb{G}_{m, \overline{K}}, \Lambda) \cong H_c^1(\mathbb{G}_{m, \overline{K}}, \Lambda)^*(-1)$  and  $H_c^1(\mathbb{G}_{m, \overline{K}}, \Lambda)$  is isomorphic to  $(i_{0*}\Lambda \oplus i_{\infty*}\Lambda)/\Lambda$ . Then the claim follows, because the zero point and the infinity point correspond to the irreducible components passing  $x_e$ .  $\square$

We put

$$H^i(\mathcal{W}'_{\overline{K}}, R^j\Psi_{\mathcal{W}_{\circ\mathbf{C}}}\mathbb{Q}_\ell) = \left( \varprojlim_{N \in \mathbb{N}} H^i(\mathcal{W}'_{\overline{K}}, R^j\Psi_{\mathcal{W}_{\circ\mathbf{C}}}\mathbb{Z}/\ell^N\mathbb{Z}) \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Then we have a spectral sequence

$$H^i(\mathcal{W}'_{\overline{K}}, R^j\Psi_{\mathcal{W}_{\circ\mathbf{C}}}\mathbb{Q}_\ell) \Rightarrow H^{i+j}(W_{\mathbf{C}}, \mathbb{Q}_\ell)$$

by [Fa, Proposition 5.9.4], because  $W$  satisfies the condition in [Fa, Proposition 5.9.4] by Proposition 3.7, where we consider  $W_{\mathbf{C}}$  as a Berkovich space. This spectral sequence gives an exact sequence

$$0 \longrightarrow H^1(\mathcal{W}'_{\overline{K}}, \mathbb{Q}_\ell) \longrightarrow H^1(W_{\mathbf{C}}, \mathbb{Q}_\ell) \longrightarrow H^0(\mathcal{W}'_{\overline{K}}, R^1\Psi_{\mathcal{W}_{\circ\mathbf{C}}}\mathbb{Q}_\ell) \longrightarrow H^2(\mathcal{W}'_{\overline{K}}, \mathbb{Q}_\ell) \quad (4.3)$$

by (4.1).

Let  $\widetilde{\Gamma}_{\mathcal{W}'}$  be a graph obtained by adding one new vertex to  $\Gamma_{\mathcal{W}'}$  and joining the new vertex with all vertices in  $\mathcal{V}(\Gamma_{\mathcal{W}'}) \setminus \mathcal{V}(\Gamma_{\mathcal{W}})$ .

**Lemma 4.2.** *There is a canonical isomorphism*

$$h: H_1(\widetilde{\Gamma}_{\mathcal{W}'}, \mathbb{Q}_\ell)(-1) \xrightarrow{\sim} \mathrm{Ker} \left( H^0(\mathcal{W}'_{\overline{K}}, R^1\Psi_{\mathcal{W}_{\circ\mathbf{C}}}\mathbb{Q}_\ell) \longrightarrow H^2(\mathcal{W}'_{\overline{K}}, \mathbb{Q}_\ell) \right).$$

*Proof.* By the isomorphism (4.2), we have a canonical isomorphism

$$H^0(\mathcal{W}'_{\overline{K}}, R^1\Psi_{\mathcal{W}_{\circ\mathbf{C}}}\mathbb{Q}_\ell) \cong H^0 \left( \mathcal{W}'_{\overline{K}}, \left( \bigoplus_{v \in \mathcal{V}(\Gamma_{\mathcal{W}'})} \pi_{v*}\mathbb{Q}_\ell \right) / \mathbb{Q}_\ell \right)^* (-1).$$



Under the identification by this isomorphism, we explain the definition of  $h$ . For  $e \in \mathcal{E}(\Gamma_{\mathcal{W}'})$ , let  $P_e$  be a point of  $Y_{s(e)} \cap Y_{t(e)}$  that correspond to  $e$ , and define  $c_e$  as an element

$$(0, 1) \in ((\pi_{s(e)*} \mathbb{Q}_\ell \oplus \pi_{t(e)*} \mathbb{Q}_\ell) / \mathbb{Q}_\ell)_{P_e} \subset H^0 \left( \mathcal{W}_{\bar{k}}, \left( \bigoplus_{v \in \mathcal{V}(\Gamma_{\mathcal{W}'})} \pi_{v*} \mathbb{Q}_\ell \right) / \mathbb{Q}_\ell \right).$$

Then we define  $\mathbb{Q}_\ell$ -linear map  $h$  so that

$$h(r)(c_e) = \langle r, e \rangle$$

for  $r \in H_1(\tilde{\Gamma}_{\mathcal{W}}, \mathbb{Q}_\ell) \subset E(\tilde{\Gamma}_{\mathcal{W}}, \mathbb{Q}_\ell)$  and  $e \in \mathcal{E}(\Gamma_{\mathcal{W}'})$ , where  $e$  is considered as an element of  $E(\Gamma_{\mathcal{W}'}, \mathbb{Q}_\ell) \subset E(\tilde{\Gamma}_{\mathcal{W}}, \mathbb{Q}_\ell)$ . We can easily check that  $h$  gives an isomorphism.  $\square$

**Proposition 4.3.** *We have two exact sequences*

$$\begin{aligned} 0 \longrightarrow H^1(\mathcal{W}_{\bar{k}}, \mathbb{Q}_\ell) &\longrightarrow H^1(W_{\mathbf{C}}, \mathbb{Q}_\ell) \longrightarrow H_1(\tilde{\Gamma}_{\mathcal{W}}, \mathbb{Q}_\ell)(-1) \longrightarrow 0, \\ 0 \longrightarrow H^1(\Gamma_{\mathcal{W}}, \mathbb{Q}_\ell) &\longrightarrow H^1(\mathcal{W}_{\bar{k}}, \mathbb{Q}_\ell) \longrightarrow \bigoplus_{v \in \mathcal{V}(\Gamma_{\mathcal{W}'})} H^1(\tilde{Y}_v, \mathbb{Q}_\ell) \longrightarrow 0. \end{aligned}$$

*Proof.* The first exact sequence follows from (4.3) and Lemma 4.2.

By a short exact sequence

$$0 \longrightarrow \mathbb{Q}_\ell \longrightarrow \bigoplus_{v \in \mathcal{V}(\Gamma_{\mathcal{W}'})} \pi_{v*} \mathbb{Q}_\ell \longrightarrow \left( \bigoplus_{v \in \mathcal{V}(\Gamma_{\mathcal{W}'})} \pi_{v*} \mathbb{Q}_\ell \right) / \mathbb{Q}_\ell \longrightarrow 0,$$

we have an exact sequence

$$\begin{aligned} \bigoplus_{v \in \mathcal{V}(\Gamma_{\mathcal{W}'})} H^0(\tilde{Y}_v, \mathbb{Q}_\ell) &\longrightarrow H^0 \left( \mathcal{W}_{\bar{k}}, \left( \bigoplus_{v \in \mathcal{V}(\Gamma_{\mathcal{W}'})} \pi_{v*} \mathbb{Q}_\ell \right) / \mathbb{Q}_\ell \right) \\ &\longrightarrow H^1(\mathcal{W}_{\bar{k}}, \mathbb{Q}_\ell) \longrightarrow \bigoplus_{v \in \mathcal{V}(\Gamma_{\mathcal{W}'})} H^1(\tilde{Y}_v, \mathbb{Q}_\ell) \longrightarrow 0. \end{aligned}$$

This exact sequence and a canonical isomorphism

$$\mathrm{Coker} \left( \bigoplus_{v \in \mathcal{V}(\Gamma_{\mathcal{W}'})} H^0(\tilde{Y}_v, \mathbb{Q}_\ell) \longrightarrow H^0 \left( \mathcal{W}_{\bar{k}}, \left( \bigoplus_{v \in \mathcal{V}(\Gamma_{\mathcal{W}'})} \pi_{v*} \mathbb{Q}_\ell \right) / \mathbb{Q}_\ell \right) \right) \cong H^1(\Gamma_{\mathcal{W}'}, \mathbb{Q}_\ell)$$

gives the second exact sequence, because  $H^1(\Gamma_{\mathcal{W}'}, \mathbb{Q}_\ell) \cong H^1(\Gamma_{\mathcal{W}}, \mathbb{Q}_\ell)$  and  $H^1(\tilde{Y}_v, \mathbb{Q}_\ell) = 0$  for  $v \in \mathcal{V}(\Gamma_{\mathcal{W}'}) \setminus \mathcal{V}(\Gamma_{\mathcal{W}})$ .  $\square$

**Remark 4.4.** *Proposition 4.3 can be considered as an explicit description of a part of the weight spectral sequence of  $W_{\mathbf{C}}$ .*

## 5 Pushforward and pullback

Let  $W_1$  and  $W_2$  be connected wide open rigid curves with semi-stable coverings  $\mathcal{S}_1 = \{(U_{1,i}, U_{1,i}^u) \mid i \in I_1\}$  and  $\mathcal{S}_2 = \{(U_{2,i}, U_{2,i}^u) \mid i \in I_2\}$  respectively. All the construction in the section 3 applies to  $W_1$  and  $W_2$ , and the subscript 1 and 2 mean that it is constructed from  $W_1$  and  $W_2$  respectively.

**Definition 5.1.** We say that a finite flat morphism  $f: W_1 \rightarrow W_2$  is compatible with semi-stable coverings if, for any  $i_1 \in I_1$ , there is  $i_2 \in I_2$  such that  $f(U_{1,i_1}^u) = U_{2,i_2}^u$  and  $f(U_{1,i_1} \setminus U_{1,i_1}^u) = U_{2,i_2} \setminus U_{2,i_2}^u$ .

Let  $f: W_1 \rightarrow W_2$  be a finite flat morphism of degree  $n$  that is compatible with semi-stable coverings. The morphism  $f$  induces a finite morphism  $\hat{f}: \mathcal{W}_1 \rightarrow \mathcal{W}_2$  by Proposition 3.6. Further,  $\hat{f}$  induces  $\hat{f}_{\bar{k}}: \mathcal{W}_{1,\bar{k}} \rightarrow \mathcal{W}_{2,\bar{k}}$  and a finite flat morphism  $\phi_f: \Gamma_{\mathcal{W}_1'} \rightarrow \Gamma_{\mathcal{W}_2'}$  of degree  $n$ . This induces a finite flat morphism  $\phi_f: \Gamma_{\mathcal{W}_1} \rightarrow \Gamma_{\mathcal{W}_2}$  of degree  $n$ . The morphism  $\phi_f$  naturally extends to a finite flat morphism  $\tilde{\phi}_f: \tilde{\Gamma}_{\mathcal{W}_1} \rightarrow \tilde{\Gamma}_{\mathcal{W}_2}$  of degree  $n$ .

In the remaining of this section,  $j = 1, 2$ . We put  $\mathcal{V}_j = \text{Spf } \mathcal{O}_K[[S_j, T_j]]/(S_j T_j - c_j)$  for some  $c_j \neq 0 \in \mathfrak{m}_K$ . Let  $Y_j$  and  $Y_j'$  be closed subscheme of the geometric closed fiber  $\mathcal{V}_{j,\bar{k}}$  defined by  $T_j = 0$  and  $S_j = 0$  respectively. We note that  $Y_j = Y_j' = \mathcal{V}_{j,\bar{k}}$ . We put  $\mathcal{V}_{j,\mathcal{O}_C} = \mathcal{V}_j \hat{\otimes}_{\mathcal{O}_K} \mathcal{O}_C$  and

$$H^0(\mathcal{V}_{j,\bar{k}}, R^1 \Psi_{\mathcal{V}_{j,\mathcal{O}_C}} \mathbb{Q}_\ell) = \left( \varprojlim_{N \in \mathbb{N}} H^0(\mathcal{V}_{j,\bar{k}}, R^1 \Psi_{\mathcal{V}_{j,\mathcal{O}_C}} \mathbb{Z}/\ell^N \mathbb{Z}) \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Then we have a canonical isomorphism

$$H^0(\mathcal{V}_{j,\bar{k}}, R^1 \Psi_{\mathcal{V}_{j,\mathcal{O}_C}} \mathbb{Q}_\ell) \cong H^0(\mathcal{V}_{j,\bar{k}}, (i_{Y_j*} \mathbb{Q}_\ell \oplus i_{Y_j'*} \mathbb{Q}_\ell)/\mathbb{Q}_\ell)^*(-1) \quad (5.1)$$

where  $i_{Y_j}: Y_j \rightarrow \mathcal{V}_{j,\bar{k}}$  and  $i_{Y_j'}: Y_j' \rightarrow \mathcal{V}_{j,\bar{k}}$  are identity morphisms. Under the identification (5.1), we define  $\gamma_j \in H^0(\mathcal{V}_{j,\bar{k}}, R^1 \Psi_{\mathcal{V}_{j,\mathcal{O}_C}} \mathbb{Q}_\ell)$  by

$$\gamma_j(a, a') = a - a' \text{ for } (a, a') \in H^0(\mathcal{V}_{j,\bar{k}}, (i_{Y_j*} \mathbb{Q}_\ell \oplus i_{Y_j'*} \mathbb{Q}_\ell)/\mathbb{Q}_\ell).$$

**Lemma 5.2.** Let  $V_j$  be the open annulus associated to  $\mathcal{V}_j$ . Let  $g: \mathcal{V}_1 \rightarrow \mathcal{V}_2$  be a finite morphism such that the induced morphism  $g: V_1 \rightarrow V_2$  is a finite flat morphism of degree  $m$ . We assume that  $g: \mathcal{V}_1 \rightarrow \mathcal{V}_2$  induces a finite morphism  $\text{Spf } k[[S_1]] \rightarrow \text{Spf } k[[S_2]]$ . Let

$$g_*: H^0(\mathcal{V}_{j,\bar{k}}, R^1 \Psi_{\mathcal{V}_{1,\mathcal{O}_C}} \mathbb{Q}_\ell) \longrightarrow H^0(\mathcal{V}_{j,\bar{k}}, R^1 \Psi_{\mathcal{V}_{2,\mathcal{O}_C}} \mathbb{Q}_\ell)$$

be the pushforward by  $g$ , and

$$g^*: H^0(\mathcal{V}_{j,\bar{k}}, R^1 \Psi_{\mathcal{V}_{2,\mathcal{O}_C}} \mathbb{Q}_\ell) \longrightarrow H^0(\mathcal{V}_{j,\bar{k}}, R^1 \Psi_{\mathcal{V}_{1,\mathcal{O}_C}} \mathbb{Q}_\ell)$$

be the pullback by  $g$ . Then  $g_*(\gamma_1) = \gamma_2$  and  $g^*(\gamma_2) = m\gamma_1$ .

*Proof.* We have a canonical isomorphism

$$H^1(V_{j,\mathcal{C}}, \mathbb{Q}_\ell) \cong H^0(\mathcal{V}_{j,\bar{k}}, R^1 \Psi_{\mathcal{V}_{j,\mathcal{O}_C}} \mathbb{Q}_\ell) \quad (5.2)$$

by arguments in the section 3. Let  $N$  be a positive integer. Using a long exact sequence obtained from

$$0 \longrightarrow (\mathbb{Z}/\ell^N \mathbb{Z})(1) \longrightarrow \mathcal{O}_{V_{j,\mathcal{C}}}^\times \xrightarrow{\ell^N} \mathcal{O}_{V_{j,\mathcal{C}}}^\times \longrightarrow 0,$$

we have an injection  $\mathcal{O}_{V_{j,\mathcal{C}}}^\times / (\mathcal{O}_{V_{j,\mathcal{C}}}^\times)^{\ell^N} \longrightarrow H^1(V_{j,\mathcal{C}}, (\mathbb{Z}/\ell^N \mathbb{Z})(1))$ . We consider a scheme  $\mathcal{X}_j = \text{Spec } \mathcal{O}_K[S_j, T_j]/(S_j T_j - c_j)$ . By considering a commutative diagram

$$\begin{array}{ccccc} \mathcal{O}_{\mathcal{X}_{j,\bar{K}}}^\times / (\mathcal{O}_{\mathcal{X}_{j,\bar{K}}}^\times)^{\ell^N}(-1) & \xrightarrow{\sim} & H^1(\mathcal{X}_{j,\bar{K}}, (\mathbb{Z}/\ell^N \mathbb{Z})) & \xrightarrow{\sim} & H^0(\mathcal{X}_{j,\bar{K}}, R^1 \Psi_{\mathcal{X}_{j,\bar{K}}}(\mathbb{Z}/\ell^N \mathbb{Z})) \\ \downarrow & & \downarrow & & \downarrow \wr \\ \mathcal{O}_{V_{j,\mathcal{C}}}^\times / (\mathcal{O}_{V_{j,\mathcal{C}}}^\times)^{\ell^N}(-1) & \hookrightarrow & H^1(V_{j,\mathcal{C}}, (\mathbb{Z}/\ell^N \mathbb{Z})) & \xrightarrow{\sim} & H^0(\mathcal{V}_{j,\bar{k}}, R^1 \Psi_{\mathcal{V}_{j,\mathcal{O}_C}}(\mathbb{Z}/\ell^N \mathbb{Z})) \end{array}$$

where the right vertical arrow is isomorphism by [Be2, Theorem 3.1]. From this commutative diagram, we see that

$$\mathcal{O}_{\mathcal{X}_{j,\overline{\mathbf{C}}}}^\times / (\mathcal{O}_{\mathcal{X}_{j,\overline{\mathbf{C}}}}^\times)^{\ell^N}(-1) \longrightarrow H^1(V_{j,\mathbf{C}}, (\mathbb{Z}/\ell^N\mathbb{Z}))$$

is an isomorphism.

We show that  $g^*$  induces

$$\mathcal{O}_{\mathcal{X}_{2,\overline{\mathbf{C}}}}^\times / (\mathcal{O}_{\mathcal{X}_{2,\overline{\mathbf{C}}}}^\times)^{\ell^N} \rightarrow \mathcal{O}_{\mathcal{X}_{1,\overline{\mathbf{C}}}}^\times / (\mathcal{O}_{\mathcal{X}_{1,\overline{\mathbf{C}}}}^\times)^{\ell^N}; S_2 \mapsto S_1^m.$$

For  $0 < t < 1$ , we define a circle  $C_t$  over  $\mathbf{C}$  by

$$C_t(\mathbf{C}) = \{x \in \mathbf{C} \mid |x| = t\}.$$

For  $c \in \overline{K}$  such that  $|c| < 1$  and  $|c|$  is sufficiently close to 1,  $g$  induce a finite flat morphism

$$C_{|c|} \cong \mathrm{Spf} \mathbf{C}\langle X_1, X_1^{-1} \rangle \longrightarrow C_{m|c|} \cong \mathrm{Spf} \mathbf{C}\langle X_2, X_2^{-1} \rangle$$

of degree  $m$  and  $g^*(X_2) = c'X_1^m g'(X_1)$ , where  $X_1 = S_1/c$ ,  $X_2 = S_2/c^m$ ,  $c' \neq 0 \in K(c)$  and

$$g'(X_1) = 1 + \sum_{k \neq 0} a_k X_1^k \in K(c)\langle X_1, X_1^{-1} \rangle$$

for  $a_k \in K(c)$  such that  $|a_k| < 1$ . We take such  $c \in \overline{K}$ . Then  $g'(X_1)$  is  $\ell$ -divisible in  $K(c)\langle X_1, X_1^{-1} \rangle$ . This shows the claim and that  $g^*(\gamma_2) = m\gamma_1$ . Then we have  $g_*(\gamma_1) = \gamma_2$ , because  $g_* \circ g^* = m$  on  $H^1(V_{2,\mathbf{C}}, \mathbb{Q}_\ell)$  by [Be, Theorem 5.4.1.(d)].  $\square$

**Theorem 5.3.** *Let  $f: W_1 \rightarrow W_2$  be a finite flat morphism that is compatible with semi-stable coverings. Then  $f$  induces the following commutative diagrams:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(\mathcal{W}_{1,\overline{k}}, \mathbb{Q}_\ell) & \longrightarrow & H^1(W_{1,\mathbf{C}}, \mathbb{Q}_\ell) & \longrightarrow & H_1(\tilde{\Gamma}_{\mathcal{W}_1}, \mathbb{Q}_\ell)(-1) \longrightarrow 0 \\ & & \hat{f}_{\overline{k}*} \downarrow & & f_{\mathbf{C}*} \downarrow & & \tilde{\phi}_{f*} \downarrow \\ 0 & \longrightarrow & H^1(\mathcal{W}_{2,\overline{k}}, \mathbb{Q}_\ell) & \longrightarrow & H^1(W_{2,\mathbf{C}}, \mathbb{Q}_\ell) & \longrightarrow & H_1(\tilde{\Gamma}_{\mathcal{W}_2}, \mathbb{Q}_\ell)(-1) \longrightarrow 0, \\ \\ 0 & \longrightarrow & H^1(\Gamma_{\mathcal{W}_1}, \mathbb{Q}_\ell) & \longrightarrow & H^1(\mathcal{W}_{1,\overline{k}}, \mathbb{Q}_\ell) & \longrightarrow & \bigoplus_{v \in \mathcal{V}(\Gamma_{\mathcal{W}_1})} H^1(\tilde{Y}_v, \mathbb{Q}_\ell) \longrightarrow 0 \\ & & \phi_{f*} \downarrow & & \hat{f}_{\overline{k}*} \downarrow & & \hat{f}_{\overline{k}*} \downarrow \\ 0 & \longrightarrow & H^1(\Gamma_{\mathcal{W}_2}, \mathbb{Q}_\ell) & \longrightarrow & H^1(\mathcal{W}_{2,\overline{k}}, \mathbb{Q}_\ell) & \longrightarrow & \bigoplus_{v \in \mathcal{V}(\Gamma_{\mathcal{W}_2})} H^1(\tilde{Y}_v, \mathbb{Q}_\ell) \longrightarrow 0, \\ \\ 0 & \longrightarrow & H^1(\mathcal{W}_{2,\overline{k}}, \mathbb{Q}_\ell) & \longrightarrow & H^1(W_{2,\mathbf{C}}, \mathbb{Q}_\ell) & \longrightarrow & H_1(\tilde{\Gamma}_{\mathcal{W}_2}, \mathbb{Q}_\ell)(-1) \longrightarrow 0 \\ & & \hat{f}_{\overline{k}}^* \downarrow & & f_{\mathbf{C}}^* \downarrow & & \tilde{\phi}_f^* \downarrow \\ 0 & \longrightarrow & H^1(\mathcal{W}_{1,\overline{k}}, \mathbb{Q}_\ell) & \longrightarrow & H^1(W_{1,\mathbf{C}}, \mathbb{Q}_\ell) & \longrightarrow & H_1(\tilde{\Gamma}_{\mathcal{W}_1}, \mathbb{Q}_\ell)(-1) \longrightarrow 0, \\ \\ 0 & \longrightarrow & H^1(\Gamma_{\mathcal{W}_2}, \mathbb{Q}_\ell) & \longrightarrow & H^1(\mathcal{W}_{2,\overline{k}}, \mathbb{Q}_\ell) & \longrightarrow & \bigoplus_{v \in \mathcal{V}(\Gamma_{\mathcal{W}_2})} H^1(\tilde{Y}_v, \mathbb{Q}_\ell) \longrightarrow 0 \\ & & \phi_f^* \downarrow & & \hat{f}_{\overline{k}}^* \downarrow & & \hat{f}_{\overline{k}}^* \downarrow \\ 0 & \longrightarrow & H^1(\Gamma_{\mathcal{W}_1}, \mathbb{Q}_\ell) & \longrightarrow & H^1(\mathcal{W}_{1,\overline{k}}, \mathbb{Q}_\ell) & \longrightarrow & \bigoplus_{v \in \mathcal{V}(\Gamma_{\mathcal{W}_1})} H^1(\tilde{Y}_v, \mathbb{Q}_\ell) \longrightarrow 0. \end{array}$$

*Proof.* We can easily check the commutativity of the second and fourth diagrams from the construction of the short exact sequences. The commutativity of the former half of the first and third diagrams are trivial.

For  $e \in \mathcal{E}(\Gamma_{\mathcal{W}'_j})$ , we define

$$\gamma_e \in H^0\left(\mathcal{W}_{j,\bar{k}}, \left((\pi_{t(e)*} \mathbb{Q}_\ell \oplus \pi_{s(e)*} \mathbb{Q}_\ell) / \mathbb{Q}_\ell\right)^*(-1)\right) \subset H^0\left(\mathcal{W}_{j,\bar{k}}, \left(\left(\bigoplus_{v \in \mathcal{V}(\Gamma_{\mathcal{W}'_j})} \pi_{v*} \mathbb{Q}_\ell\right) / \mathbb{Q}_\ell\right)^*(-1)\right)$$

by  $\gamma_e(a, a') = a - a'$  for  $(a, a') \in H^0(\mathcal{W}_{j,\bar{k}}, (\pi_{t(e)*} \mathbb{Q}_\ell \oplus \pi_{s(e)*} \mathbb{Q}_\ell) / \mathbb{Q}_\ell)$ , and consider  $\gamma_e$  as an element of  $H^0(\mathcal{W}_{j,\bar{k}}, R^1\Psi_{\mathcal{W}_j, \circ_{\mathbb{C}}} \mathbb{Q}_\ell)$  by a canonical isomorphism

$$R^1\Psi_{\mathcal{W}_j, \circ_{\mathbb{C}}} \mathbb{Q}_\ell \cong \left(\left(\bigoplus_{v \in \mathcal{V}(\Gamma_{\mathcal{W}'_j})} \pi_{v*} \mathbb{Q}_\ell\right) / \mathbb{Q}_\ell\right)^*(-1).$$

We define  $\gamma_e = 0$  for  $e \in \mathcal{E}(\tilde{\Gamma}_{\mathcal{W}_j}) \setminus \mathcal{E}(\Gamma_{\mathcal{W}'_j})$ .

We show the commutativity of the latter half of the first diagram. We consider a cycle  $R = e_1 \cdots e_m$  of  $\tilde{\Gamma}_{\mathcal{W}_1}$  as an element of  $H_1(\tilde{\Gamma}_{\mathcal{W}_1}, \mathbb{Q}_\ell)$ . Then it corresponds to  $\sum_{i=1}^m \gamma_{e_i} \in H^0(\mathcal{W}_{1,\bar{k}}, R^1\Psi_{\mathcal{W}_1, \circ_{\mathbb{C}}} \mathbb{Q}_\ell)$ . We have

$$f_*\left(\sum_{i=1}^m \gamma_{e_i}\right) = \sum_{i=1}^m \gamma_{\tilde{\phi}_{f,E}(e_i)} \in H^0(\mathcal{W}_{2,\bar{k}}, R^1\Psi_{\mathcal{W}_2, \circ_{\mathbb{C}}} \mathbb{Q}_\ell)$$

by Lemma 5.2, and this corresponds to  $\tilde{\phi}_{f*}(R)$ .

We show the commutativity of the latter half of the third diagram. We consider a cycle  $R' = e'_1 \cdots e'_m$  of  $\tilde{\Gamma}_{\mathcal{W}_2}$  as an element of  $H_1(\tilde{\Gamma}_{\mathcal{W}_2}, \mathbb{Q}_\ell)$ . Then it corresponds to  $\sum_{i=1}^m \gamma_{e'_i} \in H^0(\mathcal{W}_{2,\bar{k}}, R^1\Psi_{\mathcal{W}_2, \circ_{\mathbb{C}}} \mathbb{Q}_\ell)$ . We have

$$f^*\left(\sum_{i=1}^m \gamma_{e'_i}\right) = \sum_{i=1}^m \sum_{e \in \tilde{\phi}_{f,E}^{-1}(e'_i)} n_e \gamma_e \in H^0(\mathcal{W}_{1,\bar{k}}, R^1\Psi_{\mathcal{W}_1, \circ_{\mathbb{C}}} \mathbb{Q}_\ell)$$

by Lemma 5.2, and this corresponds to  $\tilde{\phi}_f^*(R')$ .  $\square$

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